Indian Statistical Institute, Bangalore

B. Math.(Hons.) I Year, First Semester

Mid-Sem Examination

Analysis -I

Time: 3 hours   September 11, 2009   Instructor: Pl.Muthuramalingam

May marks you can get is 30

1. If $f, g : [a, b] \longrightarrow R$ are continuous functions then $h = \min\ \{f, g\}$ is also a continuous function. [2]

2. Let $f(x) = a_0 + a_1\ x + a_2\ x^2 + \cdots$ where $a_0, a_1, a_2 \cdots$ are real nuberes. Let $\{limsup\ |a_n|^{\frac{1}{n}}\}^{-1} = R_0$

   (i) Find R_0 when $a_n = \frac{1}{\angle n}$. [1]

   (ii) Give example of $a_0,\ a_1,\ a_2 \cdots$ such that $R_o = 1$, for $|x| = R_0$, the series $\sum a_n\ x^n$ is divrgent for $x = 1$ and convergent for $x = -1$. [2]

3. Let $I_1 \supset I_2 \supset \cdots$ be a sequence of closed intervals with length $I_r \longrightarrow 0$ as $r \longrightarrow \infty$. If $x_j, \epsilon\ I_j$ then show that the sequence $\{x_1,\ x_2,\ x_3, \cdots\}$ is a cauchy sequence. [2]

4. Let $x_1,\ x_2,\ x_3, \cdots$ be bounded sequence of reals with $x_j \geq 0$. If every subsequence of x_n has a [further] subsequence converging to 0 show that $x_n \longrightarrow 0$. [3]

5. (a) Let $a_n > 0, \sum a_n^2 \quad < \infty,\ \partial > \frac{1}{2}$. Then shwo that $\sum_1^\infty \frac{a_n}{n^\partial}$ exists. [2]

   (b) Let $\mathbb{B} > \frac{1}{2}$. Show that $\sum\ b_n < \infty$ where $b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^{\mathbb{B}}}$. [2]

   (c) If $|x| < |$ show that $x^n \longrightarrow 0$ as $n \longrightarrow \infty$. [1]

6. (a) Show that $f : R \longrightarrow R$ given by $f(x) = x^2$ is not uniformly continuous. [2]

   (b) Let f be as in (a). Show that if $x_1, x_2, \cdots$ is a cauchy sequence than $f(x_1), f(x_2), \cdots$ is a cauchy sequence. [2]

   (c) Give an example of uniformly continuous functions g_1, g_2 such that the product $g_1\ g_2$ is not uniformly continuous and prove your claim.[1]

   (d) Let $g : J \longrightarrow R$ be uniformly continuous.Show that if $x_1, x_2, \cdots$ is a cauchy sequence in J, then $g(x_1), \cdots$ is a cauchy sequence. [2]

   (e) Let $h : (0, 1] \longrightarrow R$ be uniformly continuous. If $y_n \varepsilon (0, 1]$ and $y_n \longrightarrow 0$ then $h(y_n)$ is convergent and the limit is independent of the sequence $y_1, y_2, \cdots$ (converging to 0). [3]

(f) Let $k : J \longrightarrow R$ be a continuous, differentiable function and the derivative be bounded and continuous. Show that k is uniformly continuous. Here J is a bounded or unbounded interval. [2]

7. If $a_1, a_2, \cdots$ is a sequence of reals with $\sum |a_n| < \infty$, then $\sum a_n$ exists. [2]

8. Let $a_1, a_2, a_3, \cdots$ be a sequence of reals. $s_n = a_1 + a_2 + a_3 + a_n$. Assume that the sequence s_{3n} is convergent. Then $\sum a_n$ exists $\Leftrightarrow$ $a_r \longrightarrow 0$ as $r \longrightarrow \infty$. [2]

9. Let $a_n > 0$ and $\sum\limits_1^\infty a_n$ be divergent. Let $b_n = \frac{a_n}{1+a_n}$. Show that $\sum\limits_1^\infty b_n$ is divergent. [4]

10. Let $a_n, b_n > 0 \quad a_n \longrightarrow a$ with $a \neq o$. Show that lim sup $(a_n b_n) =$ a lim supb_n. [3]

11. Let $x_1, x_2, \cdots$ be a bounded sequence and $\mathbb{B} =$ limsup x_n. If $\varepsilon > 0$, show that $(\mathbb{B} + \varepsilon, \infty)$ can have only finitely many of the $x_1, x_2, x_3, \cdots$ [3]